

Bounds of the remainder in a combinatorial central limit theorem

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May 8, 2014

Abstract

We derive new bounds of the remainder in a combinatorial central limit theorem without assumptions on independence and existence of moments of summands. For independent random variables our theorems imply Esseen and Berry-Esseen type inequalities, some other new bounds and a combinatorial central limit theorem in the case of infinite variations.

AMS 2000 subject classification: 60F05

Key words: combinatorial central limit theorem, Berry-Esseen inequality, Esseen inequality

1 Introduction

Let $\|X_{ij}\|$ be a $n \times n$ matrix of independent random variables and $\vec{\pi} = (\pi(1), \pi(2), \dots, \pi(n))$ be a random permutation of $1, 2, \dots, n$, independent with X_{ij} . Assume that $\vec{\pi}$ has the uniform distribution on the set all such permutations. Denote

$$S_n = \sum_{i=1}^n X_{i\pi(i)}.$$

First results on asymptotical normality of S_n were obtained for $P(X_{ij} = c_{ij}) = 1$, $1 \leq i, j \leq n$, in Wald and Wolfowitz (1944). They found sufficient conditions for that when $c_{ij} = a_i b_j$. Noether (1949) proved that these conditions maybe relaxed. Hoeffding (1951) considered general case of c_{ij} and obtained a combinatorial central limit theorem

(CLT). Further results on the combinatorial CLT were obtained by Motoo (1957) and Kolchin and Chistyakov (1973).

Later investigations have been turned from limit theorems to non-asymptotic results similar to Berry–Esseen and Esseen inequalities in classical theory of summing of independent random variables. Von Bahr (1976) and Ho and Chen (1978) derived bounds for the remainder in a combinatorial CLT in the case of non-degenerated X_{ij} . Botlthausen (1984) obtained Esseen type inequality for the remainder for degenerated X_{ij} . The constant was not be specified in the last paper. Further results of this type may be found in Goldstein (2005) and Chen, Goldstein and Shao (2011). They contain explicit constants in the inequalities. For non-degenerated X_{ij} , Esseen type inequalities were stated by Neammanee and Suntornchost (2005), Neammanee and Rattanawong (2009) and Chen and Fang (2012). These inequalities were obtained for X_{ij} with finite third moments by an application of Stein method. At the same time, it is known that the Berry–Esseen and Esseen inequalities maybe generalized to random variables without third moments. This techniques for sums of independent random variables may be found in Petrov (1995), for example. Applying similar techniques, Frolov (2014) obtained Esseen type bounds for the remainder in a combinatorial CLT for X_{ij} with finite variations without third moments.

In this paper, we obtain new bounds for the remainder in a combinatorial CLT without moment assumptions. We also prove a general result in which there are no independence assumptions. In the case of independent random variables, our new results generalize those in Frolov (2014). Moreover, our results yield a combinatorial CLT for random variables without second moments. In our example, the summands belong to the domain of attraction of the normal law.

2 Results

Let $\|X_{ij}\|$ be a $n \times n$ matrix of random variables and $\vec{\pi} = (\pi(1), \pi(2), \dots, \pi(n))$ be a random permutation of $1, 2, \dots, n$, where $n \geq 2$. Note that we do not suppose the independence of random variables under consideration.

Denote

$$S_n = \sum_{i=1}^n X_{i\pi(i)}.$$

For real a_n and $b_n > 0$, put

$$\Delta_n = \sup_{x \in \mathbb{R}} \left| P \left(\frac{S_n - a_n}{b_n} < x \right) - \Phi(x) \right|,$$

where $\Phi(x)$ is the standard normal distribution function.

Let $\|\mu_{ij}\|$ be a $n \times n$ matrix of real numbers and $\|t_{ij}\|$ be a $n \times n$ matrix with $0 < t_{ij} \leq +\infty$, where $n \geq 2$. For $1 \leq i, j \leq n$, put

$$\bar{X}_{ij} = (X_{ij} - \mu_{ij})I\{|X_{ij} - \mu_{ij}| < t_{ij}\},$$

where $I\{\cdot\}$ denotes the indicator of the event in brackets. Denote

$$\bar{S}_n = \sum_{i=1}^n \mu_{i\pi(i)} + \sum_{i=1}^n \bar{X}_{i\pi(i)}, \quad \bar{e}_n = E\bar{S}_n, \quad \bar{B}_n = D\bar{S}_n = E\bar{S}_n^2 - (\bar{e}_n)^2,$$

and

$$\bar{\Delta}_n = \sup_{x \in \mathbb{R}} \left| P\left(\frac{\bar{S}_n - \bar{e}_n}{\sqrt{\bar{B}_n}} < x\right) - \Phi(x) \right|.$$

For all i and j put $p_{ij} = P(\pi(i) = j)$ and

$$q_{ij} = \begin{cases} P(|X_{ij} - \mu_{ij}| \geq t_{ij} | \pi(i) = j), & \text{if } p_{ij} > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Our first result is as follows.

Theorem 1. *The following inequality holds*

$$\Delta_n \leq \bar{\Delta}_n + \Psi_n + \Theta_n + \Upsilon_n, \quad (1)$$

where

$$\Psi_n = \sum_{i,j=1}^n q_{ij} p_{ij}, \quad \Theta_n = \frac{|a_n - \bar{e}_n|}{\sqrt{2\pi} \sqrt{\bar{B}_n}}, \quad \Upsilon_n = \frac{1}{\sqrt{2\pi} e} \max\left(\frac{\sqrt{\bar{B}_n}}{b_n} - 1, \frac{b_n}{\sqrt{\bar{B}_n}} - 1\right).$$

This result is an analogue of Theorem 5.9 in Petrov (1995) for sums of random variables.

We now turn to the main case when random variables X_{ij} are independent and permutation $\vec{\pi}$ is independent from summands and has the uniform distribution.

For every $n \times n$ matrix $\|m_{ij}\|$, put

$$m_{i.} = \frac{1}{n} \sum_{j=1}^n m_{ij}, \quad m_{.j} = \frac{1}{n} \sum_{i=1}^n m_{ij}, \quad m_{..} = \frac{1}{n^2} \sum_{i,j=1}^n m_{ij}, \quad m_{ij}^* = m_{ij} - m_{i.} - m_{.j} + m_{..}$$

for all i and j .

Denote $\bar{a}_{ij} = E\bar{X}_{ij}$ and $\bar{\sigma}_{ij}^2 = D\bar{X}_{ij}$ for $1 \leq i, j \leq n$. It is not difficult to check that

$$\bar{e}_n = n(\bar{a}_{..} + \mu_{..}), \quad \bar{B}_n = \frac{1}{n-1} \sum_{i,j=1}^n (\mu_{ij}^* + \bar{a}_{ij}^*)^2 + \frac{1}{n} \sum_{i,j=1}^n \bar{\sigma}_{ij}^2.$$

Moreover, in this case,

$$\Psi_n = \frac{1}{n} \sum_{i,j=1}^n P(|X_{ij} - \mu_{ij}| \geq t_{ij}),$$

and Theorem 1 has the following form.

Theorem 2. Assume that random variables X_{ij} are independent and permutation $\vec{\pi}$ is independent with X_{ij} . Suppose that $\vec{\pi}$ has the uniform distribution on the set of all permutation of $1, 2, \dots, n$.

Then the following inequality holds

$$\Delta_n \leq \bar{\Delta}_n + \frac{1}{n} \sum_{i,j=1}^n P(|X_{ij} - \mu_{ij}| \geq t_{ij}) + \Theta_n + \Upsilon_n. \quad (2)$$

There are no moment assumption in Theorems 1 and 2. We now consider the case of finite means.

Assume that $EX_{ij} = c_{ij}$ and

$$c_{i.} = c_{.j} = 0, \quad (3)$$

for all $1 \leq i, j \leq n$. Note that this property of the matrix $\|EX_{ij}\|$ plays in a combinatorial CLT the same role that the centering at mean of summands does in CLT.

Condition (3) implies that $ES_n = 0$ and, therefore, we take $a_n = 0$.

In the sequel, we also put $t_{ij} = b_n$ for all $1 \leq i, j \leq n$.

Theorem 3. Assume that the conditions of theorem 2 are satisfied, relation (3) holds and $\mu_{i.} = \mu_{.j} = 0$ for all $1 \leq i, j \leq n$.

Then there exists an absolute positive constant A such that

$$\Delta_n \leq \frac{A}{n\bar{B}_n^{3/2}} \sum_{i,j=1}^n (|\mu_{ij}|^3 + E|\bar{X}_{ij}|^3) + \frac{1}{n} \sum_{i,j=1}^n P(|X_{ij} - \mu_{ij}| \geq b_n) + \Theta_n + \Upsilon_n. \quad (4)$$

Note that we assume no moment conditions in Theorem 3 besides existence of means.

Theorem 3 contains many known results and allows to derive new bounds of remainder in a combinatorial CLT.

We start with the case of finite variations of random variables X_{ij} , in which Theorem 3 yields the following result.

Theorem 4. Assume that the conditions of Theorem 3 hold and $DX_{ij} = \sigma_{ij}^2$. Put

$$B_n = DS_n = \frac{1}{n-1} \sum_{i,j=1}^n c_{ij}^2 + \frac{1}{n} \sum_{i,j=1}^n \sigma_{ij}^2,$$

Then there exists an absolute positive constant A such that

$$\Delta_n \leq A(C_n + \Lambda_n + L_n), \quad (5)$$

where

$$C_n = \frac{1}{nB_n^{3/2}} \sum_{i,j=1}^n |\mu_{ij}|^3, \quad \Lambda_n = \frac{1}{nB_n} \sum_{i,j=1}^n \alpha_{ij}, \quad L_n = \frac{1}{nB_n^{3/2}} \sum_{i,j=1}^n \beta_{ij},$$

$\alpha_{ij} = E(X_{ij} - \mu_{ij})^2 I\{|X_{ij} - \mu_{ij}| \geq \sqrt{B_n}\}$ and $\beta_{ij} = E|X_{ij} - \mu_{ij}|^3 I\{|X_{ij} - \mu_{ij}| < \sqrt{B_n}\}$ for $1 \leq i, j \leq n$.

We would like to mention that constants A are different in our theorems. Of course, one can find them as function of the constant in inequality (4). The last constant becomes from bounds in a combinatorial CLT for summands with third moments. Unfortunately, this constant is large now and, therefore, we do not give exact expressions here.

Theorem 4 is a generalization of Theorems 1 and 4 from Frolov (2014), where the cases $\mu_{ij} = 0$ and $\mu_{ij} = c_{ij}$ for all $1 \leq i, j \leq n$ have been considered. In the same way as in Frolov (2014), we arrive at the following result.

Theorem 5. *Assume that the conditions of Theorem 4 hold. Let $g(x)$ be a positive, even function such that $g(x)$ and $x/g(x)$ are non-decreasing for $x > 0$. Suppose that $g_{ij} = E(X_{ij} - \mu_{ij})^2 g(X_{ij} - \mu_{ij}) < \infty$ for $1 \leq i, j \leq n$.*

Then there exists an absolute positive constant A such that

$$\Delta_n \leq A \left(\frac{1}{B_n^{3/2} n} \sum_{i,j=1}^n |\mu_{ij}|^3 + \frac{1}{B_n g(\sqrt{B_n}) n} \sum_{i,j=1}^n g_{ij} \right). \quad (6)$$

Theorem 5 includes as partial cases Theorems 2 and 5 from Frolov (2014), where $\mu_{ij} = 0$ and $\mu_{ij} = c_{ij}$ for all $1 \leq i, j \leq n$, correspondingly. For $g(x) = |x|^{2+\delta}$, $\delta \in (0, 1]$, we get the following result from Theorem 5.

Theorem 6. *Assume that the conditions of Theorem 4 hold.*

Then there exists an absolute positive constant A such that

$$\Delta_n \leq A \left(\frac{1}{B_n^{3/2} n} \sum_{i,j=1}^n |\mu_{ij}|^3 + \frac{1}{B_n^{1+\delta/2} n} \sum_{i,j=1}^n E|X_{ij} - \mu_{ij}|^{2+\delta} \right),$$

where $\delta \in (0, 1]$.

Theorem 6 improves Theorems 3 and 6 from Frolov (2014), where $\mu_{ij} = 0$ and $\mu_{ij} = c_{ij}$ for all $1 \leq i, j \leq n$, correspondingly.

Note that Theorems 4 and 6 imply a combinatorial CLT under Lyapunov and Lindeberg type conditions, correspondingly.

Theorems 5 and 6 may be applied to $-X_{ij}$ as well. Nevertheless, one can derive further results from Theorem 4 under non-symmetric conditions on distributions of X_{ij} by a method from Frolov (2014). Making use of this method, one can obtain bounds in terms of sums of $E|X_{ij}|^{2+\delta_{ij}}$ or some other moments depending on i and j .

Let us turn to the case of infinite variations. In this case, Theorem 3 also gives new results.

It is clear that we would like to put $b_n = \sqrt{\bar{B}_n}$ in this case. The problem is that \bar{B}_n depends on b_n . Then consider the relation $b_n = \sqrt{\bar{B}_n}$ as an equation to determine b_n . Let us show how it works on an example.

Assume that X_{ij} have the same distribution with the density

$$p(x) = \begin{cases} |x|^{-3}, & \text{if } |x| > 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $c_{ij} = \bar{a}_{ij} = 0$,

$$\bar{\sigma}_{ij}^2 = \int_{|x| < b_n} x^2 p(x) dx = 2 \log b_n$$

for all $1 \leq i, j \leq n$ and

$$\bar{B}_n = \frac{1}{n} \sum_{i,j=1}^n \bar{\sigma}_{ij}^2 = 2 \log b_n.$$

It follows that the equation $b_n = \sqrt{\bar{B}_n}$ turns to

$$b_n = \sqrt{2n \log b_n}.$$

It is not difficult to check that

$$b_n \sim \sqrt{n \log n} \quad \text{as } n \rightarrow \infty.$$

We have

$$P(|X_{ij}| \geq b_n) = \int_{|x| \geq b_n} p(x) dx = \frac{1}{b_n^2} \sim \frac{1}{n \log n} \quad \text{as } n \rightarrow \infty.$$

Moreover,

$$E|\bar{X}_{ij}|^3 = \int_{|x| < b_n} x^3 p(x) dx = 2(b_n - 1).$$

Relations $b_n = \sqrt{\bar{B}_n}$ and $a_{..} = 0$ imply that $\Upsilon_n = 0$ and $\Theta_n = 0$, correspondingly. It follows from (4) that

$$\Delta_n \leq A \frac{n(b_n - 1)}{b_n^3} + \frac{n}{b_n^2} = O\left(\frac{1}{\log n}\right) \quad \text{as } n \rightarrow \infty.$$

It yields that Theorem 3 gives a combinatorial CLT with a bound for a rate of convergence. Moreover, norming $\sqrt{n \log n}$ is determined in a similar way as for distributions from the domain of attraction of the standard normal law in usual CLT.

We now state a variant of a combinatorial CLT that follows from Theorem 3. We consider the case $\mu_{ij} = c_{ij}$.

Theorem 7. Let $\{\|X_{nij}\|; 1 \leq i, j \leq n, n = 2, 3, \dots\}$ be a sequence of $n \times n$ matrix of independent random variables with $EX_{nij} = c_{nij}$ and $\vec{\pi}_n = (\pi(1), \pi(2), \dots, \pi(n))$ be random permutations of $1, 2, \dots, n$, independent with X_{nij} . Assume that $\vec{\pi}_n$ has the uniform distribution on the set all permutations of $1, 2, \dots, n$ for $n = 2, 3, \dots$. Denote

$$S_n = \sum_{i=1}^n X_{ni\pi_n(i)}.$$

Assume that $c_{ni.} = c_{n.j} = 0$ for all i, j and n .

Let $\{b_n\}$ be a sequence of positive constants. Put $\bar{X}_{nij} = (X_{nij} - c_{nij})I\{|X_{nij} - c_{nij}| < b_n\}$, $\bar{a}_{nij} = E\bar{X}_{nij}$, $\bar{\sigma}_{nij}^2 = D\bar{X}_{nij}$ for all i, j and n . Denote

$$\bar{B}_n = \frac{1}{n-1} \sum_{i,j=1}^n (c_{nij} + \bar{a}_{nij} - \bar{a}_{ni.} - \bar{a}_{n.j} + \bar{a}_{n..})^2 + \frac{1}{n} \sum_{i,j=1}^n \bar{\sigma}_{nij}^2.$$

Assume that the following conditions hold:

- 1) $\frac{1}{b_n^3 n} \sum_{i,j=1}^n |c_{nij}|^3 \rightarrow 0$ as $n \rightarrow \infty$,
- 2) $\frac{1}{n} \sum_{i,j=1}^n P(|X_{nij} - c_{nij}| \geq \varepsilon b_n) \rightarrow 0$ as $n \rightarrow \infty$ for every fixed $\varepsilon > 0$,
- 3) $\frac{\bar{B}_n}{b_n^2} \rightarrow 1$ as $n \rightarrow \infty$,
- 4) $\frac{1}{b_n n} \sum_{i,j=1}^n |\bar{a}_{nij}| \rightarrow 0$ as $n \rightarrow \infty$.

Then

$$\sup_{x \in \mathbb{R}} \left| P\left(\frac{S_n}{b_n} < x\right) - \Phi(x) \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

3 Proofs

Proof of Theorem 1. Put $p_n = b_n/\sqrt{\bar{B}_n}$, $q_n = (a_n - \bar{e}_n)/\sqrt{\bar{B}_n}$,

$$\begin{aligned} \Delta_{n1} &= \sup_{x \in \mathbb{R}} \left| P\left(\frac{S_n - a_n}{b_n} < x\right) - P\left(\frac{\bar{S}_n - a_n}{b_n} < x\right) \right|, \\ \Delta_{n2} &= \sup_{x \in \mathbb{R}} \left| P\left(\frac{\bar{S}_n - \bar{e}_n}{\sqrt{\bar{B}_n}} < p_n x + q_n\right) - \Phi(p_n x + q_n) \right|, \\ \Delta_{n3} &= \sup_{x \in \mathbb{R}} |\Phi(p_n x + q_n) - \Phi(x)|. \end{aligned}$$

We have

$$\Delta_n \leq \Delta_{n1} + \Delta_{n2} + \Delta_{n3}.$$

It is clear that $\Delta_{n2} = \bar{\Delta}_n$ and, therefore, we will estimate Δ_{n1} and Δ_{n2} .
Since

$$S_n = \bar{S}_n + \sum_{i=1}^n (X_{i\pi(i)} - \mu_{i\pi(i)}) I \{ |X_{i\pi(i)} - \mu_{i\pi(i)}| \geq t_{i\pi(i)} \},$$

we have

$$\{S_n < x\} \subset \{\bar{S}_n < x\} \cup \bigcup_{i=1}^n \{|X_{i\pi(i)} - \mu_{i\pi(i)}| \geq t_{i\pi(i)}\}.$$

It follows that

$$\begin{aligned} P(S_n < x) &\leq P(\bar{S}_n < x) + \sum_{i=1}^n P(|X_{i\pi(i)} - \mu_{i\pi(i)}| \geq t_{i\pi(i)}) \\ &= P(\bar{S}_n < x) + \sum_{i=1}^n \sum_{j=1}^n P(|X_{i\pi(i)} - \mu_{i\pi(i)}| \geq t_{i\pi(i)}, \pi(i) = j) \\ &= P(\bar{S}_n < x) + \sum_{i=1}^n \sum_{j=1}^n p_{ij} q_{ij} = P(\bar{S}_n < x) + \Psi_n. \end{aligned}$$

From the other hand

$$\{\bar{S}_n < x\} \subset \{S_n < x\} \cup \bigcup_{i=1}^n \{|X_{i\pi(i)} - \mu_{i\pi(i)}| \geq t_{i\pi(i)}\},$$

which yields that

$$P(\bar{S}_n < x) \leq P(S_n < x) + \Psi_n.$$

It follows that

$$\Delta_{n1} \leq \Psi_n.$$

The following result is a corollary of Lemma 5.2 in Petrov (1995).

Lemma 1. *For every real $p > 0$ and q the following inequality holds*

$$\sup_{x \in \mathbb{R}} |\Phi(px + q) - \Phi(x)| \leq \frac{|q|}{\sqrt{2\pi}} + \frac{1}{\sqrt{2\pi}e} \max\left(p - 1, \frac{1}{p} - 1\right).$$

By Lemma 1 we get $\Delta_{n3} \leq \Theta_n + \Upsilon_n$. This finishes the proof. \square

Proof of Theorem 3. We need the following known results (see, for example, Chen and Fang (2012)).

Theorem A. Let $\|Y_{ij}\|$ be $n \times n$ matrix of independent random variables such that $EY_{ij} = \nu_{ij}$, $DY_{ij} = v_{ij}$ and $E|Y_{ij}|^3 < \infty$ for all i and j . Let $\vec{\pi} = (\pi(1), \pi(2), \dots, \pi(n))$ be a random permutation of $1, 2, \dots, n$ with uniform distribution on the set of all permutations. Assume that $\vec{\pi}$ and random variables Y_{ij} are independent.

Then there exist an absolute constant A such that

$$\sup_{x \in \mathbb{R}} \left| P \left(\frac{V_n - n\nu_{..}}{\sigma} < x \right) - \Phi(x) \right| \leq \frac{A}{n\sigma^{3/2}} \sum_{i,j=1}^n E|Y_{ij} - \nu_{i.} - \nu_{.j} + \nu_{..}|^3,$$

where

$$V_n = \sum_{i=1}^n Y_{i\pi(i)}, \quad \sigma^2 = DV_n = \frac{1}{n-1} \sum_{i,j=1}^n (\nu_{ij}^*)^2 + \frac{1}{n-1} \sum_{i,j=1}^n v_{ij}.$$

By Theorem A with $Y_{ij} = \mu_{ij} + \bar{X}_{ij}$, we have that

$$\bar{\Delta}_n \leq \frac{A}{n\bar{B}_n^{3/2}} \sum_{i,j=1}^n E|\mu_{ij} + \bar{X}_{ij} - \bar{a}_{i.} - \bar{a}_{.j} + \bar{a}_{..}|^3.$$

By the Hölder inequality we, get

$$\bar{\Delta}_n \leq \frac{25A}{n\bar{B}_n^{3/2}} \sum_{i,j=1}^n (|\mu_{ij}|^3 + E|\bar{X}_{ij}|^3 + |\bar{a}_{i.}|^3 + |\bar{a}_{.j}|^3 + |\bar{a}_{..}|^3).$$

Making use of the Lyapunov inequality, we obtain that $|\bar{a}_{ij}| \leq (E|\bar{X}_{ij}|^3)^{1/3}$ for all i and j . Applying again the Hölder inequality, we write

$$|\bar{a}_{i.}|^3 = \frac{1}{n^3} \left| \sum_{j=1}^n \bar{a}_{ij} \right|^3 \leq \frac{1}{n^3} \left(\sum_{j=1}^n |\bar{a}_{ij}| \right)^3 \leq \frac{1}{n} \sum_{j=1}^n |\bar{a}_{ij}|^3 \leq \frac{1}{n} \sum_{j=1}^n E|\bar{X}_{ij}|^3.$$

It follows that

$$\sum_{i,j=1}^n |\bar{a}_{i.}|^3 \leq \sum_{i,j=1}^n E|\bar{X}_{ij}|^3.$$

In the same way, we arrive at

$$\sum_{i,j=1}^n |\bar{a}_{.j}|^3 \leq \sum_{i,j=1}^n E|\bar{X}_{ij}|^3.$$

Further, an application of the Hölder inequality yields that

$$|\bar{a}_{..}|^3 = \frac{1}{n^6} \left| \sum_{i,j=1}^n \bar{a}_{ij} \right|^3 \leq \frac{1}{n^6} \left(\sum_{i,j=1}^n |\bar{a}_{ij}| \right)^3 \leq \frac{1}{n^2} \sum_{i,j=1}^n |\bar{a}_{ij}|^3 \leq \frac{1}{n^2} \sum_{i,j=1}^n E|\bar{X}_{ij}|^3.$$

The latter inequality implies that

$$\sum_{i,j=1}^n |\bar{a}_{..}|^3 \leq \sum_{i,j=1}^n E|\bar{X}_{ij}|^3.$$

It follows that

$$\bar{\Delta}_n \leq \frac{25A}{n\bar{B}_n^{3/2}} \sum_{i,j=1}^n (|\mu_{ij}|^3 + 4E|\bar{X}_{ij}|^3).$$

This bound and inequality (2) yield (4) and Theorems is proved. \square

Lemma 2. *Assume that the conditions of Theorem 4 hold. Then there exists an absolute constant A' such that*

$$\left| 1 - \frac{\sqrt{\bar{B}_n}}{\sqrt{B_n}} \right| \leq A'(C_n + \Lambda_n).$$

Proof of Lemma 2. Assume that $B_n = 1$. Then

$$\bar{X}_{ij} = (X_{ij} - \mu_{ij})I\{|X_{ij} - \mu_{ij}| < 1\}.$$

Put

$$\hat{X}_{ij} = (X_{ij} - \mu_{ij})I\{|X_{ij} - \mu_{ij}| \geq 1\}.$$

We have

$$1 - \bar{B}_n = B_n - \bar{B}_n = \frac{1}{n} \sum_{i,j=1}^n (\sigma_{ij}^2 - \bar{\sigma}_{ij}^2) + \frac{1}{n-1} \sum_{i,j=1}^n (c_{ij}^2 - (\mu_{ij} + \bar{a}_{ij} - \bar{a}_{i.} - \bar{a}_{.j} + \bar{a}_{..})^2).$$

Note that for all i and j ,

$$c_{ij} = \mu_{ij} + \bar{a}_{ij} + E\hat{X}_{ij}. \quad (7)$$

It follows that

$$\sigma_{ij}^2 - \bar{\sigma}_{ij}^2 = E(X_{ij} - \mu_{ij})^2 - (c_{ij} - \mu_{ij})^2 - E\bar{X}_{ij}^2 + \bar{a}_{ij}^2 = E\hat{X}_{ij}^2 - 2\bar{a}_{ij}E\hat{X}_{ij} - (E\hat{X}_{ij})^2,$$

for all i and j . Taking into account that $|\bar{a}_{ij}| < 1$, we have

$$|\sigma_{ij}^2 - \bar{\sigma}_{ij}^2| \leq 4E\hat{X}_{ij}^2.$$

Then

$$\frac{1}{n} \sum_{i,j=1}^n (\sigma_{ij}^2 - \bar{\sigma}_{ij}^2) \leq 4\Lambda_n. \quad (8)$$

Further, applying of (7) implies that for all i and j ,

$$\begin{aligned} & c_{ij}^2 - (\mu_{ij} + \bar{a}_{ij} - \bar{a}_{i.} - \bar{a}_{.j} + \bar{a}_{..})^2 \\ &= 2(\mu_{ij} + \bar{a}_{ij})E\hat{X}_{ij} + (E\hat{X}_{ij})^2 + 2(\mu_{ij} + \bar{a}_{ij})(\bar{a}_{i.} + \bar{a}_{.j} - \bar{a}_{..}) - (\bar{a}_{i.} + \bar{a}_{.j} - \bar{a}_{..})^2. \end{aligned}$$

Note that

$$\sum_{i,j=1}^n \mu_{ij} \bar{a}_{i.} = n \sum_{i=1}^n \mu_{i.} \bar{a}_{i.} = 0,$$

and, similarly,

$$\sum_{i,j=1}^n \mu_{ij} \bar{a}_{.j} = 0, \quad \sum_{i,j=1}^n \mu_{ij} \bar{a}_{..} = 0.$$

It follows that

$$\begin{aligned} & \sum_{i,j=1}^n (c_{ij}^2 - (\mu_{ij} + \bar{a}_{ij} - \bar{a}_{i.} - \bar{a}_{.j} + \bar{a}_{..})^2) \\ &= \sum_{i,j=1}^n \left(2(\mu_{ij} + \bar{a}_{ij}) E \hat{X}_{ij} + (E \hat{X}_{ij})^2 + 2\bar{a}_{ij}(\bar{a}_{i.} + \bar{a}_{.j} - \bar{a}_{..}) - (\bar{a}_{i.} + \bar{a}_{.j} - \bar{a}_{..})^2 \right). \end{aligned}$$

Making use of the Hölder inequality, we have

$$(\bar{a}_{i.} + \bar{a}_{.j} - \bar{a}_{..})^2 \leq 3(\bar{a}_{i.}^2 + \bar{a}_{.j}^2 + \bar{a}_{..}^2). \quad (9)$$

We write

$$\begin{aligned} \sum_{i,j=1}^n \bar{a}_{i.}^2 &= n \sum_{i=1}^n \bar{a}_{i.}^2 = \frac{1}{n} \sum_{i=1}^n \left(\sum_{j=1}^n (c_{ij} - \mu_{ij} - E \hat{X}_{ij}) \right)^2 \\ &= \frac{1}{n} \sum_{i=1}^n \left(n c_{i.} - n \mu_{i.} - \sum_{j=1}^n E \hat{X}_{ij} \right)^2 = \frac{1}{n} \sum_{i=1}^n \left(\sum_{j=1}^n E \hat{X}_{ij} \right)^2 \\ &\leq \sum_{i,j=1}^n (E \hat{X}_{ij})^2 \leq \sum_{i,j=1}^n E \hat{X}_{ij}^2 = n \Lambda_n. \end{aligned} \quad (10)$$

We obtain in the same way that

$$\sum_{i,j=1}^n \bar{a}_{.j}^2 \leq n \Lambda_n, \quad \sum_{i,j=1}^n \bar{a}_{..}^2 \leq n \Lambda_n. \quad (11)$$

Further, we get by (10) that

$$\sum_{i,j=1}^n \bar{a}_{ij} \bar{a}_{i.} = n \sum_{i=1}^n \bar{a}_{i.}^2 \leq n \Lambda_n. \quad (12)$$

It follows from (11) in the same way that

$$\sum_{i,j=1}^n \bar{a}_{ij} \bar{a}_{.j} \leq n\Lambda_n, \quad \sum_{i,j=1}^n \bar{a}_{ij} \bar{a}_{..} \leq n\Lambda_n. \quad (13)$$

Taking into account that $xy \leq x^3/3 + 2y^{3/2}/3$ for all non-negative x and y , we get

$$|\mu_{ij} E \hat{X}_{ij}| \leq \frac{1}{3} |\mu_{ij}|^3 + \frac{2}{3} |E \hat{X}_{ij}|^{3/2} \leq \frac{1}{3} |\mu_{ij}|^3 + \frac{2}{3} E \hat{X}_{ij}^2$$

for all i and j . Hence

$$\sum_{i,j=1}^n |\mu_{ij} E \hat{X}_{ij}| \leq \frac{nC_n}{3} + \frac{2n\Lambda_n}{3}. \quad (14)$$

Since $|\bar{a}_{ij}| < 1$ for all i and j , we conclude that

$$\sum_{i,j=1}^n |\bar{a}_{ij} E \hat{X}_{ij}| \leq \sum_{i,j=1}^n E \hat{X}_{ij}^2 = n\Lambda_n. \quad (15)$$

It follows from (9)–(15) that

$$\frac{1}{n-1} \sum_{i,j=1}^n (c_{ij}^2 - (\mu_{ij} + \bar{a}_{ij} - \bar{a}_{i.} - \bar{a}_{.j} + \bar{a}_{..})^2) \leq \frac{2n}{3(n-1)} C_n + \left(\frac{4n}{3(n-1)} + \frac{18n}{(n-1)} \right) \Lambda_n.$$

The last inequality and (8) yield that

$$|B_n - \bar{B}_n| = |1 - \bar{B}_n| \leq \frac{2n}{3(n-1)} C_n + \left(\frac{4n}{3(n-1)} + \frac{18n}{(n-1)} + 4 \right) \Lambda_n \leq C_n + 43\Lambda_n,$$

and Lemma 2 is proved for $B_n = 1$. If $B_n \neq 1$, then we apply the latter inequality to $X_{ij}/\sqrt{B_n}$, $c_{ij}/\sqrt{B_n}$ and $\mu_{ij}/\sqrt{B_n}$. \square

Proof of Theorem 4. Assume that $B_n = 1$.

If $\sqrt{\bar{B}_n} \leq 1/2$, then by Lemma 2

$$\Delta_n \leq 1 \leq 2|1 - \sqrt{\bar{B}_n}| \leq 2A'(C_n + \Lambda_n).$$

It yields (5) for $B_n = 1$ in this case.

Assume now that $\sqrt{\bar{B}_n} \geq 1/2$. If $\sqrt{\bar{B}_n} > 1$, then we have by Lemma 2 that

$$\Upsilon_n = \frac{1}{\sqrt{2\pi e}} (\sqrt{\bar{B}_n} - 1) \leq \frac{A'}{\sqrt{2\pi e}} (C_n + \Lambda_n).$$

For $\sqrt{B_n} < 1$, we get by Lemma 2 that

$$\Upsilon_n = \frac{1}{\sqrt{2\pi e}} \left(\frac{1}{\sqrt{B_n}} - 1 \right) \leq \frac{1}{\sqrt{2\pi e}} 2(1 - \sqrt{B_n}) \leq \frac{2A'}{\sqrt{2\pi e}} (C_n + \Lambda_n).$$

Note that

$$\bar{X}_{ij} = (X_{ij} - \mu_{ij})I\{|X_{ij} - \mu_{ij}| < 1\}.$$

Put again

$$\hat{X}_{ij} = (X_{ij} - \mu_{ij})I\{|X_{ij} - \mu_{ij}| \geq 1\}.$$

It is clear that

$$\frac{1}{n} \sum_{i,j=1}^n P(|X_{ij} - \mu_{ij}| \geq 1) = \frac{1}{n} \sum_{i,j=1}^n EI\{|X_{ij} - \mu_{ij}| \geq 1\} \leq \frac{1}{n} \sum_{i,j=1}^n E\hat{X}_{ij}^2 = \Lambda_n.$$

Moreover,

$$\begin{aligned} \Theta_n &= \frac{|\bar{e}_n|}{\sqrt{2\pi}\sqrt{B_n}} = \frac{1}{\sqrt{2\pi}\sqrt{B_n}n} \left| \sum_{i,j=1}^n \bar{a}_{ij} \right| = \frac{1}{\sqrt{2\pi}\sqrt{B_n}n} \left| \sum_{i,j=1}^n E\hat{X}_{ij} \right| \\ &\leq \frac{1}{\sqrt{2\pi}\sqrt{B_n}n} \sum_{i,j=1}^n E\hat{X}_{ij}^2 = \frac{1}{\sqrt{2\pi}\sqrt{B_n}} \Lambda_n \leq \frac{\Lambda_n}{\sqrt{\pi}}. \end{aligned}$$

These bounds imply by (4) that

$$\Delta_n \leq 2^{3/2} A(C_n + L_n) + \frac{2A'}{\sqrt{2\pi e}} (C_n + \Lambda_n) + \Lambda_n + \frac{\Lambda_n}{\sqrt{\pi}}.$$

This inequality yields (5) for $B_n = 1$.

If $B_n \neq 1$, then we apply the result for $B_n = 1$ to $X_{ij}/\sqrt{B_n}$, $c_{ij}/\sqrt{B_n}$ and $\mu_{ij}/\sqrt{B_n}$. \square

Theorems 5 and 6 follow from Theorem 4 in the same way as in Frolov (2014). Details are omitted.

Proof of Theorem 7. Conditions 3) and 4) imply that $\Upsilon_n \rightarrow 0$ and $\Theta_n \rightarrow 0$ as $n \rightarrow \infty$, correspondingly.

Take $\varepsilon > 0$. We have

$$\begin{aligned} E|\bar{X}_{nij}|^3 &= E|X_{nij} - c_{nij}|^3 I\{|X_{nij} - c_{nij}| < \varepsilon b_n\} + E|X_{nij} - c_{nij}|^3 I\{\varepsilon b_n \leq |X_{nij} - c_{nij}| < b_n\} \\ &\leq \varepsilon b_n E|X_{nij} - c_{nij}|^2 I\{|X_{nij} - c_{nij}| < b_n\} + b_n^3 P(|X_{nij} - c_{nij}| \geq \varepsilon b_n) \\ &= \varepsilon b_n (\bar{\sigma}_{nij}^2 + \bar{a}_{nij}^2) + b_n^3 P(|X_{nij} - c_{nij}| \geq \varepsilon b_n) \leq \varepsilon b_n \bar{\sigma}_{nij}^2 + \varepsilon b_n^2 |\bar{a}_{nij}| + b_n^3 P(|X_{nij} - c_{nij}| \geq \varepsilon b_n), \end{aligned}$$

for all i and j . Hence

$$\frac{1}{nb_n^3} \sum_{i,j=1}^n E|\bar{X}_{nij}|^3 \leq \varepsilon \frac{\bar{B}_n}{b_n^2} + \varepsilon \frac{1}{b_n n} \sum_{i,j=1}^n |\bar{a}_{nij}| + \frac{1}{n} \sum_{i,j=1}^n P(|X_{nij} - c_{nij}| \geq \varepsilon b_n).$$

This inequality and conditions 2), 3) and 4) yield that

$$\frac{1}{nb_n^3} \sum_{i,j=1}^n E|\bar{X}_{nij}|^3 \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

By inequality (4), we arrive at the desired conclusion. \square

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